Dual Simplex Algorithm

- In "primal" simplex, RHS column in always non-negative, hence basic solution is feasible at every iteration
- What if some elements of the RHS column are negative?
- In such a case, primal is infeasible
- Dual Simplex Algorithm (DSA) : addresses such a scenario
- DSA: particularly useful for re-optimizing a problem after a constraint has been added or some problem parameter has been changed (sensitivity analysis), such that a previously optimal basis is no longer feasible

Dual Simplex Algorithm : Concept

- At each iteration of "primal" simplex:
  - always maintain primal feasibility (RHS≥0)
  - drive towards primal optimality (in other words dual feasibility), i.e., coefficients of variables in (-z) row≤0
  - corresponding dual is always infeasible
- At each iteration of "dual" simplex
  - always maintain primal optimality, i.e., coefficients of variables in (-z) row≤0
  - In other words, always maintain dual feasibility
  - drive towards primal feasibility (RHS≥0)
  - terminate when primal feasibility is attained, i.e., all elements in RHS column ≥ 0

Dual Canonical Form

- All decision variables ≥ 0
- All RHS coefficients negative (only difference with "primal" simplex)
- All constraints, except non-negativity stated as equalities
- Isolate one decision variable from each constraint with +1 coefficient, which does not appear in any other constraint and appears with a zero coefficient in the objective function

Procedure of Dual Simplex method

- Convert any functional constraint in ≥ form to ≤ form by multiplying both sides by –1
- Introduce slack variables as needed
- Identify leaving variable
  - variable to leave is the basic variable associated with the constraint with most negative RHS value
- Row corresponding to leaving variable called "pivot row"

- Perform ratio test to identify entering variable
  - Pick all negative coefficients in pivot row (a_{lj})
  - Let x_l be leaving variable
  - Compute the ratios (c_j/ a_{lj}) where all $a_{lj} < 0$
- Column(variable) that gives smallest ratio enters basis
- Identify pivot element (as in "primal" simplex)
- Divide it by itself to make it 1
- Make other elements in the column of the pivot element = 0 by performing row operations
- Continue till all elements in RHS column become ≥ 0

Example 1

- Consider the following LP

$$\text{Max } -3x_1 - 4x_2$$
$$\text{s.t. } -2x_1 + x_2 \le -2$$
$$x_1 + 2x_2 \ge 4$$
$$x_1, x_2 \ge 0$$

Step 1: Multiply second constraint by –1 to convert to $\le$ Form

$$\text{Max } -3x_1 - 4x_2$$
$$\text{s.t. } -2x_1 + x_2 \le -2$$
$$-x_1 - 2x_2 \le -4$$
$$x_1, x_2 \ge 0$$

Example 1-Contd…

Step 2: Add slack variables, convert into dual canonical form

$$\text{Max } -3x_1 - 4x_2$$

$$\text{s.t. } -2x_1 + x_2 + x_3 = -2$$

$$-x_1 - 2x_2 + x_4 = -4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical form shown below

$$-2x_1 + x_2 + x_3 + 0x_4 = -2$$

$$-x_1 - 2x_2 + 0x_3 + x_4 = -4$$

$$-3x_1 - 4x_2 + 0x_3 + 0x_4 = 0$$

1st Tableau

Pivot Row

Leaves

Basic Vars	RHS	x_1	x_2	x_3	x_4
x_3	-2	-2	1	1	0
x_4	-4	-1	-2	0	1
(-z)	0	-3	-4	0	0
Ratio		-3/-1=3	-4/-2=2		

Enters

Pivot Element, make it 1 and other elements in column of x_2 =0 by row operations

- x_4 Leaves

2^{nd} Tableau

Basic Vars	RHS	x_1	x_2	x_3	x_4
x_3	-4	-5/2	0	1	½
x_2	2	½	1	0	-½
(-z)	8	-1	0	0	-2
Ratio		-1/(-5/2)=2/5			

x_3 leaves, x_1 enters

3rd Tableau

Basic Vars	RHS	x_1	x_2	x_3	x_4
x_1	8/5	1	0	-2/5	-1/5
x_2	6/5	0	1	1/5	-2/5
(-z)	48/5	0	0	-2/5	-11/5
Ratio					

Note:

All RHS elements are now ≥ 0

Hence we are done

Optimal solution: z=-48/5, x_1=8/5, x_2=6/5

Points to Note

- In “primal” simplex, first identify entering variable, then leaving variable
- In “dual” simplex, first identify leaving variable, then entering variable
- At each iteration, all elements of (-z) row ≤ 0
- At each iteration, the dual to the original problem is always feasible
  - Verify this by writing the dual to the original problem
  - Obtain values form dual multipliers from each tableau
    - At each iteration, dual multipliers = values of slacks in (z) row, e.g. at 2nd iteration, dual multipliers are 0 and 2

Example 2

$$\text{Max } -x_1-2x_2$$

s.t.

$$-x_1+2x_2-x_3 \leq -2$$

$$-2x_1-x_2+x_3 \leq -6$$

$$x_1,\ x_2,\ x_3 \geq 0$$

1^{st} Tableau

Basic Vars	RHS	x_1	x_2	x_3	x_4	x_5
x_4	-4	-1	2	-1	1	0
x_5	-6	-2	-1	1	0	1
(-z)	0	-1	-2	0	0	0
Ratio		-1/-2=1/2	-2/-1=2			

x_5 leaves, x_1 enters

2^{nd} Tableau

Basic Vars	RHS	x_1	x_2	x_3	x_4	x_5
x_4	-1	0	5/2	-3/2	1	-½
x_1	3	1	½	-½	0	-½
(-z)	3	0	3/2	-½	0	-½
Ratio				(-1/2)/(3/2)=1/3		

x_4 leaves, x_3 enters

3rd Tableau

Basic Vars	RHS	x_1	x_2	x_3	x_4	x_5
x_3	2/3	0	-5/3	1	-2/3	1/3
x_1	10/3	1	-1/3	0	-1/3	-1/3
(-z)	10/3	0	-7/3	0	-1/3	-1/3
Ratio						

Optimal solution obtained: $z = -10/3$, $x_1=10/3, x_2=0$

Assignment

- Try yourself

Q.1 Obtain the Dual of

Maximize $z = 5x_1 + 4x_2 + 3x_3$

Subject to the constraints

$3x_1 + 2x_2 + x_3 \leq 10$, $2x_1 + x_2 + 2x_3 \leq 12$, $x_1 + x_2 + 3x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

Q.2 Solve the LPP by dual simplex method:

Maximize $z = -2x_1 - x_2$

Subject to the constraints:

$3x_1 + x_2 \geq 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \geq 3$ $x_1, x_2 \geq 0$

Q.3 Solve the LPP by dual simplex method:

Minimize $z = 2x_1 + 2x_2 + 4x_3$

Subject to the constraints:

$2x_1 + 3x_2 + 5x_3 \geq 2$, $3x_1 + x_2 + 7x_3 \leq 3$, $x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

- Thank you